Solemne 3 - Ecuaciones Diferenciales

1. Una definición de la función Gamma está dada por la integral impropia $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}dt$, $\alpha > 0$.

   a) $[0,5\,\text{Ptos}]$Demuestre que $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

   b) $[0,5\,\text{Ptos}]$Demuestre que $\mathfrak{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$. $\alpha > -1$

   c) $[0,5\,\text{Ptos}]$Use el hecho de que $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ para encontrar $\mathfrak{L}\left\{t^{\frac{1}{2}}\right\}$.

   Solución:
   a)

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha \mathrm{e}^{-t}dt =$$

$$= \left(-t^\alpha \mathrm{e}^{-t}\right)\Big|_0^\infty + \alpha\int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}dt =$$

$$= \alpha\Gamma(\alpha)$$

Donde:

$$u = t^\alpha \Rightarrow du = \alpha t^{\alpha-1}dt$$

$$dv = \mathrm{e}^{-t}dt \Rightarrow v = -\mathrm{e}^{-t}$$

$$-t^\alpha \mathrm{e}^{-t} \xrightarrow[t\to\infty]{} 0$$

b)

$$\mathfrak{L}\{t^\alpha\} = \int_0^\infty \mathrm{e}^{-st}t^\alpha dt = \int_0^\infty \mathrm{e}^{-z}\left(\frac{z}{s}\right)^\alpha \frac{dz}{s} =$$

$$= \frac{\int_0^\infty \mathrm{e}^{-z}z^\alpha dz}{s^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Donde:

$$z = st \Rightarrow dz = sdt$$

c)

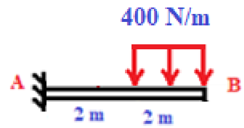
$$\mathfrak{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} =$$

$$\boxed{\mathfrak{L}\left\{t^{\frac{1}{2}}\right\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}}$$

2. [$1,5$ Ptos]La ecuación de la curva elástica $y = y(x)$ para la viga mostrada en la figura se obtiene de la ecuación diferencial de segundo orden

$$EIy'' = 800x - 2400 - 200\mathscr{U}(x-2)(x-2)^2\,,\; y(0) = y'(0) = 0$$

Donde I es el momento de inercia de la sección transversal de la viga, respecto del eje neutro y E es el módulo de elasticidad del material de la viga (módulo de Young).



Determine la expresión para EIy.

Solución:

$$EIy'' = 800x - 2400 - 200\mathscr{U}(x-2)(x-2)^2$$

Aplicamos la transformada de Laplace:

$$EI\left[s^2\mathfrak{L}\{y\} - sy(0) - y'(0)\right] = \frac{800}{s^2} - \frac{2400}{s} - 200\mathrm{e}^{-2s}\mathfrak{L}\left\{x^2\right\}$$

$$EI\left[s^2\mathfrak{L}\{y\}\right] = \frac{800}{s^2} - \frac{2400}{s} - \frac{2!\times 200\mathrm{e}^{-2s}}{s^3}$$

$$EI\mathfrak{L}\{y\} = \frac{800}{s^4} - \frac{2400}{s^3} - \frac{400\mathrm{e}^{-2s}}{s^5} \qquad [0,8 \text{ Ptos.}]$$

$$EIy = \frac{800x^3}{3!} - \frac{2400x^2}{2!} - \frac{400\mathscr{U}(x-2)(x-2)^4}{4!}$$

$$EIy = \frac{800x^3}{6} - \frac{2400x^2}{2} - \frac{50\mathscr{U}(x-2)(x-2)^4}{3}$$

$$\boxed{EIy = \frac{400x^3}{3} - 1200x^2 - \frac{50\mathscr{U}(x-2)(x-2)^4}{3}\,,\, 0 \le x \le 4m \qquad [0,7 \ \textbf{Ptos.}]}$$

3. *a*) [0, 7 Ptos]Calcule la transformada de Laplace de la siguiente función:

$$f(t) = \begin{cases} \cos(4t) & , \quad 0 \le t < \pi \\ 0 & , \quad t \ge \pi \end{cases}$$

b) [0, 8 Ptos]Use la transformada de Laplace para encontrar la solución del siguiente problema de valor inicial

$$y'' + 16y = f(t)\,,\; y(0) = 0,\; y'(0) = 1$$

Solución:
a)

$$f(t) = \begin{cases} \cos(4t) & , \quad 0 \le t < \pi \\ 0 & , \quad t \ge \pi \end{cases} \Leftrightarrow f(t) = (\mathscr{U}(t) - \mathscr{U}(t-\pi))\cos(4t) \qquad [0,3 \text{ Ptos.}]$$

$$f(t) = \mathscr{U}(t)\cos(4t) - \mathscr{U}(t-\pi)\cos(4(t-\pi)+4\pi)$$

$$\mathfrak{L}\{f(t)\} = \frac{s}{s^2+16} - e^{-\pi s}\mathfrak{L}\{\cos 4(t+\pi)\} =$$

$$\mathfrak{L}\{f(t)\} = \frac{s}{s^2+16} - e^{-\pi s}\mathfrak{L}\{\cos(4t)\cos(4\pi) - \operatorname{sen}(4t)\operatorname{sen}(4\pi)\} =$$

$$\mathfrak{L}\{f(t)\} = \frac{s}{s^2+16} - e^{-\pi s}\mathfrak{L}\{\cos(4t)\} =$$

$$\boxed{\mathfrak{L}\{f(t)\} = \frac{s}{s^2+16} - \frac{e^{-\pi s}s}{s^2+16} \qquad [0,4 \text{ Ptos.}]}$$

b) Aplicamos la transformada de Laplace:

$$s^2\mathfrak{L}\{y\} - sy(0) - y'(0) + 16\mathfrak{L}\{y\} = \mathfrak{L}\{f(t)\}$$

$$\mathfrak{L}\{y\}\left(s^2+16\right) = 1 + \frac{s}{s^2+16} - \frac{e^{-\pi s}s}{s^2+16}$$

$$\mathfrak{L}\{y\} = \frac{1}{s^2+16} + \frac{s}{\left(s^2+16\right)^2} - \frac{e^{-\pi s}s}{\left(s^2+16\right)^2} \qquad [0,5 \text{ Ptos.}]$$

$$\mathfrak{L}\{y\} = \mathfrak{L}\left\{\frac{1}{4}\operatorname{sen}(4t) + \frac{t\operatorname{sen}(4t)}{2} - \frac{1}{2}\mathscr{U}(t-\pi)(t-\pi)\operatorname{sen}4(t-\pi)\right\}$$

Entonces:

$$\boxed{y(t) = \frac{1}{4}\operatorname{sen}(4t) + \frac{t\operatorname{sen}(4t)}{2} - \frac{1}{2}\mathscr{U}(t-\pi)((t-\pi)\operatorname{sen}4(t-\pi)) \; t \ge 0 \qquad [0,3 \text{ Ptos.}]}$$

Donde:

$$\frac{s}{\left(s^2+16\right)^2} = \frac{1}{s^2+16}\frac{s}{s^2+16} = \mathfrak{L}\left\{\int_0^t \operatorname{sen}4(t-z)\cos(4z)\,dz\right\} =$$

$$= \frac{1}{2}\mathfrak{L}\left\{\int_0^t [\operatorname{sen}(4t) + \operatorname{sen}(4t-8z)]\,dz\right\} =$$

$$= \frac{1}{2}\mathfrak{L}\left\{\left[z\operatorname{sen}(4t) + \frac{1}{8}\cos(4t-8z)\right]\Big|_0^t\right\} =$$

$$= \frac{1}{2}\mathfrak{L}\left\{t\operatorname{sen}(4t) + \frac{1}{8}\cos(4t) - \frac{1}{8}\cos(4t)\right\} = \mathfrak{L}\left\{\frac{t\operatorname{sen}(4t)}{2}\right\}$$

4. *a*) [0, 7 Ptos]Halle la función f dada por la ecuación

$$f(t) = \cos t + \int_0^t e^{-z} f(t-z)\, dz$$

b) [0, 8 Ptos]Resuelva la ecuación

$$y'' - 2y' = \delta(t-2)\,,\ y(0) = 0,\ y'(0) = 1$$

Solución:
a)
Aplicamos la transformada de Laplace:

$$\mathfrak{L}\{f(t)\} = \frac{s}{s^2+1} + \frac{1}{s+1}\mathfrak{L}\{f(t)\} \qquad [0,3 \text{ Ptos.}]$$

$$\mathfrak{L}\{f(t)\}\left(1 - \frac{1}{s+1}\right) = \frac{s}{s^2+1}$$

$$\mathfrak{L}\{f(t)\}\left(\frac{s}{s+1}\right) = \frac{s}{s^2+1}$$

$$\mathfrak{L}\{f(t)\} = \frac{s+1}{s^2+1} = \mathfrak{L}\{\cos t + \operatorname{sen} t\}$$

Luego:

$$\boxed{f(t) = \cos t + \operatorname{sen} t,\ t \geq 0 \qquad [0,4 \text{ Ptos.}]}$$

b)

$$y'' - 2y' = \delta(t-2)\,,\ y(0) = 0,\ y'(0) = 1$$

Aplicamos la transformada de Laplace:

$$s^2\mathfrak{L}\{y\} - sy(0) - y'(0) - 2s\mathfrak{L}\{y\} + 2y(0) = \mathfrak{L}\{\delta(t-2)\}$$

$$s^2\mathfrak{L}\{y\} - 1 - 2s\mathfrak{L}\{y\} = e^{-2s}$$

$$\mathfrak{L}\{y\}\left(s^2 - 2s\right) = 1 + e^{-2s}$$

$$\mathfrak{L}\{y\} = \frac{1}{s(s-2)} + \frac{e^{-2s}}{s(s-2)} \qquad [0,4 \text{ Ptos.}]$$

$$\mathfrak{L}\{y\} = \mathfrak{L}\left\{\frac{1}{2}\left(e^{2t}-1\right) + \mathscr{U}(t-2)\frac{1}{2}\left(e^{2(t-2)}-1\right)\right\}$$

Luego:

$$\boxed{y(t) = \frac{1}{2}\left(e^{2t}-1\right) + \mathscr{U}(t-2)\frac{1}{2}\left(e^{2(t-2)}-1\right),\ t \geq 0 \qquad [0,4 \text{ Ptos.}]}$$

Donde:

$$\mathfrak{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \int_0^t e^{2z} dz = \frac{e^{2z}}{2}\Big|_0^t = \frac{1}{2}\left(e^{2t}-1\right)$$